On the Mean Curvature Flow for σ_k-Convex Hypersurfaces

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§1 Introduction

For $n \geq 1$, let M^n be a compact n-dimensional manifold without boundary and $F_0 : M^n \to R^{n+1}$ be a smooth immersion of M^n in R^{n+1} as a hypersurface. Recall that $M_0 = F_0(M^n)$ is said to be moved by its mean curvature, if there is a family $F(\cdot,t)$ of smooth immersions of M^n into R^{n+1} with the corresponding hypersurfaces $M_t = F(\cdot,t)(M^n)$ satisfying

$$\frac{\partial F}{\partial t}(p,t) = -H(p,t)\nu(p,t), \quad (p,t) \in M^n \times R_+, \tag{1.1}$$

$$F(p,0) = F_0(p), \qquad p \in M^n \tag{1.2}$$

where $H(p,t)$ and $\nu(p,t)$ are the mean curvature and the outward unitary vector of the hypersurface M_t at $F(p,t)$, respectively. It was proved by Huisken [H1] that there exists a $0 < T = T(M^n) \leq \infty$ such that (1.1)-(1.2) always admits a unique smooth solution for $0 < t < T$ and $\lim_{t\uparrow T} \max_{M_t} |H| = \infty$. When M_0 is a convex hypersurface, Huisken [H1] proved that M_t contracts smoothly to a point as $t \uparrow T$. Without the convexity assumption on M_0, Huisken proved in [H2] (Theorem 3.5) that if the singularity is of the Type I, i.e.

$$\max_{M_t} |A|^2 \leq \frac{C}{2(T-t)}, \tag{1.3}$$

then suitable scalings of M_t near the singularity converges smoothly to an immersed nonempty limiting hypersurface $\tilde{M}$, which satisfies the equation

$$H(x) =< x, \nu(x) >, \tag{1.4}$$

where x is the position vector, H is the mean curvature and $\nu(x)$ is the outward unit normal vector field. It was also proved by [H2] (Theorem 4.1) that compact manifolds with nonnegative mean curvature satisfying (1.4) are spheres of radius $\sqrt{n}$. For singularity of Type II, i.e.

$$\lim_{t \uparrow T}(T-t)\max_{M_t}|A|^2 = \infty. \tag{1.5}$$

Huisken and Sinestrari proved in [HS1] (Theorem 3.1) and [HS2] (theorem 1.1) that if M_0 is mean convex (i.e. $H \geq 0$) then σ_k-curvature ($2 \leq k \leq n$) of M_t satisfies, for any $0 < t < T$,

$$\sigma_k(M_t(p)) \geq -\epsilon H^k(p,t) - C_{\epsilon,n,k} \tag{1.6}$$

(see below for the definition of σ_k). Based on this key estimate, they proved in [HS2] (Theorem 4.1) that the so-called essential scaling of M_t near time T converges to a smooth mean curvature flow $\{\tilde{M}_t\}_{t\in R}$, with $\tilde{M}_t$ convex hypersurfaces. Moreover, either $\tilde{M}_t$ is strictly convex translating soliton or (up to rigid motion) $\tilde{M}_t = R^{n-k} \times \Sigma_t^k$ where Σ_t^k is a lower dimensional strictly convex soliton in R^{k+1}. For $n = 2$, it was shown by [HS1] (Corollary 4.7) that $\{\Sigma_t^1\}_{t\in R}$ is the "grim reaper" curve given by $x = -\ln\cos y + t$.

In contrast with the convex case, it is well known that the mean curvature flow (1.1)-(1.2) can develop singularities before it may shrink to a point. It is a major problem for people to study the nature of its singularity and the asymptotic behavior near the singularity. In this note, we make some effort to try to understand the structure of the singularity set at the first singular time for the initial hypersurface M_0 belonging to the class consisting of σ_k convex hypersurfaces for some $1 \leq k \leq n$. More precisely, we want to understand the limiting set M_T, which is the support of the Radon measure μ_T obtained as the limit of Radon measures μ_t, where μ_t is the area measure of M_t described as below, as $t \uparrow T$. To better describe our result, we first recall that, in addition to the above classical motion by mean curvatures, Brakke [B] has introduced the motion by its mean curvature for a family of Radon measures $\{\nu_t\}_{t\in R}$ in R^{n+1} (e.g. n-dimensional

rectifiable varifolds), which satisfies (1.1) in the weak form

$$\int \phi \, d\nu_t \leq \int \phi \, d\nu_s + \int_s^t \int (-\phi|\mathcal{H}|^2 + \mathcal{H} \cdot S^\perp \cdot D\phi) \, d\nu_t \, dt, \tag{1.7}$$

for all nonnegative $\phi \in C_0^1(R^{n+1})$ and $0 \leq s \leq t$. The reader can refer to Ilmanen [I1] for the interpretation of (1.7) and related results.

Note that if $\{M_t\}_{0 \leq t < T}$ solve (1.1)-(1.2) and we denote μ_t as the area measure of M_t for $0 \leq t < T$, then μ_t are integral varifolds of multiplicity 1 and satisfy

$$\mu_t(\phi) - \mu_s(\phi) = \int_s^t \int (-H^2\phi + D\phi \cdot \mathcal{H}) \, d\mu_t \, dt, \tag{1.8}$$

for any nonnegative $\phi \in C_0^1(R^{n+1})$ and $0 \leq s \leq t < T$, where $\phi_t(\phi) = \int \phi \, d\mu_t$ and $\mathcal{H} = -H\nu$. In particular, $\{\mu_t\}_{0 \leq t < T}$ is a motion by mean curvature by Brakke in the sense of (1.7). Moreover, (1.8) implies that there exists a nonnegative measure μ_T in R^{n+1} such that $\mu_t \to \mu_T$ as convergence of Radon measures in R^{n+1} as $t \uparrow T$. Now we extend $\{\mu_t\}_{0 \leq t \leq T}$ to $t > T$ such that $\{\mu_t\}_{0 \leq t \leq \infty}$ are a family of Radon measures moved by its mean curvatures in the sense of (1.7), whose existence is established by Brakke [B]. Our result is concerned with the properties of $M_T = \mathrm{spt}(\mu_T)$.

One of the most important facts to the family $\{\mu_t\}_{0 \leq t \leq \infty}$ is the following *monotonicity formula*, which was first discovered by Huisken [H2] for (1.1) and later obtained by Ilmanen [I2] for Brakke flows (1.7) and has played a key role in the analysis of the singularity for (1.1) and (1.7). The formula says the follows. Let $\rho_{(y,s)}$ denote the n-dimensional backward heat kernel centered at $(y, s) \in R^{n+1} \times R$ defined by

$$\rho_{(y,s)}(x, t) = (4\pi(s-t))^{-\frac{n}{2}} \exp(-\frac{|x-y|^2}{4(s-t)}), \quad x \in R^{n+1}, t < s.$$

Let $\mu = \{(\mu_t, t) : 0 \leq t \leq \infty\}$ and define, for $0 < r < s$,

$$\Theta(\mu, (y, s), r) = \int \rho_{(y,s)}(x, s-r) \, d\mu_{s-r}(x).$$

Then one has

$$\int_{s-r_2}^{s-r_1} \int \rho_{(y,s)}(x,t) |\mathcal{H}(x,t) + \frac{1}{2(s-t)} S^\perp(x,t) \cdot (x-y)|^2 \, d\mu_t(x) \, dt$$
$$\leq \Theta(\mu, (y,s), r_2) - \Theta(\mu, (y,s), r_1). \tag{1.9}$$

for any $(y,s) \in R^{n+1} \times R_+$ and $0 < r_1 < r_2 < s$, where $S(x,t) = T_x\mu_t$. A direct consequence of (1.9) is that the density function

$$\Theta(\mu,(y,s)) \equiv \lim_{r\downarrow 0} \Theta(\mu,(y,s),r)$$

exists for any $(y,s) \in R^{n+1} \times R_+$ and is upper semicontinuous. Using the upper semicontinuity, it is not difficult to prove that M_T is actually the Hausdorff limit of M_t as $t \uparrow T$ (see, Lemma 2.1 below). Our first result is

Theorem A. *The n-dimensional Hausdorff measure of M_T is finite, i.e. $H^n(M_T) < \infty$.*

By exploring the upper semicontinuity of $\Theta(\mu,\cdot)$ and extending the idea of the Federer's dimension reduction argument to the parabolic setting, White [W1] (Theorem 9) has recently obtained the stratification theorem for the support $\mathcal{M} = \{(\mathrm{spt}(\nu_t),t) : t \geq 0\}$ of Brakke flows for k-dimensional integral varifolds $\{\nu_t\}_{t\geq 0}$ in R^{n+1} $(1 \leq k \leq n)$, which roughly says that the points of $\mathcal{M}$, for which each tangent flow having its spine dimension (see [W1] for its definition) at most l, is of parabolic Hausdorff dimension at most l for all $0 \leq l \leq k+2$. Inspired by this stratification theorem by White [W1], we shall consider the stratification of the extension set M_T of the $\{\mu_t\}_{t\geq 0}$ described as above. First, note that the monotonicity of $\Theta(\mu,(y,s),\cdot)$ actually implies that $\Theta(\mu,(y,s))$ is upper semicontinuous with both of its arguments (see, e.g. [W2] theorem 2). Moreover, the uniform upper bound of $\Theta(\mu,(y,s),\cdot)$ in terms of M_0 and $(y,s) \in \mathrm{spt}(\mu)$ implies that for $x \in M_T$ if we consider the parabolic blow-up, $P_{(x,T),\lambda}(\mu)$, defined as

$$P_{(x,T),\lambda}(\mu)(\phi) = \lambda^{-n} \int \phi((x,T) + (\lambda x, \lambda^2 t))\, d\mu_t\, dt, \forall \phi \in C_0^1(R^{n+1} \times R).$$

Then for any $\lambda \to 0$ we can extract a subsequence $\lambda_i \to 0$ and a limiting Brakke flow $\tilde{\mu} \equiv \{\tilde{\mu}_t\}_{t\in R}$, which is called a tangent flow at (x,T), such that $P_{(x,T),\lambda_i}(\mu) \to \tilde{\mu}$ as convergence of Radon measures in $R^{n+1} \times R$. Moreover, as shown by [I1] and [W1] that $\tilde{\mu}$ is backwardly self-similar, i.e., $P_{(0,0),\lambda}(\tilde{\mu}|_{t\leq 0}) = \tilde{\mu}|_{t\leq 0}$, and

$$\Theta(\tilde{\mu},(0,0)) = \Theta(\mu,(x,T)) \geq \Theta(\tilde{\mu},z), \forall z = (y,s) \in R^{n+1} \times R.$$

It was also proved by [W1] that the set $V(\tilde{\mu}) \equiv \{x \in R^{n+1} : \Theta(\tilde{\mu}, (x, 0)) = \Theta(\tilde{\mu}, (0, 0))\}$ is a vector subspace of R^{n+1}. Denote $\dim(\tilde{\mu}) = \dim(V(\tilde{\mu}))$. Then, White's stratification theorem yields

Proposition B ([W1]). *Assume that the Brakke flow $\{\mu_t\}_{t\in R_+}$ is given as above (i.e. $\{\mu_t\}_{0\le t<T}$ coincides with the smooth mean curvature flow $\{M_t\}_{0\le t<T}$ of (1.1)-(1.2)). Then*

$$M_T = \Sigma_0 \cup (\Sigma_1 \setminus \Sigma_0) \cup \cdots (\Sigma_n \setminus \Sigma_{n-1}), \tag{1.10}$$

where

$$\Sigma_i \equiv \{x \in M_T : dim(\tilde{\mu}) \le i, \ \text{for any tangent flow } \tilde{\mu} \text{ at } (x, T)\}$$

for $0 \le i \le n$. Moreover, $dim_H(\Sigma_i) \le i$, for $0 \le i \le n$.

Note that the original proof of White [W1], which is in the nature of parabolic type, can be modified to prove this proposition, we would like to give a slightly different proof of it, which is in the Euclidean nature, in §3.

Now we turn our attention to the σ_k convex case of mean curvature flows. First, we recall the definition of σ_k curvatures for hypersurfaces in R^{n+1} (see also [HS2]).

Definition 1.1. For a closed hypersurface $M \subset R^{n+1}$, let $-\infty < \lambda_1(x) \le \lambda_2(x) \le \cdots \le \lambda_n(x) < \infty$ be the principal curvatures of M at $x \in M$. For $1 \le k \le n$, the σ_k curvatures of M at x is defined by

$$\sigma_k^M(x) = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1}(x)\cdots\lambda_{i_k}(x).$$

Note that the σ_1^M curvature is nothing but the mean curvature of M, σ_2^M is the scalar curvature of M, and σ_n^M is the Guassian curvature of M.

Definition 1.2. For $1 \le k \le n$. A closed hypersurface $M \subset R^{n+1}$ is called σ_k convex (respectively, strictly σ_k convex) if $\min_{x\in M}\sigma_k^M(x) \ge 0$ (respectively, $\min_{x\in M}\sigma_k^M(x) > 0$). In particular, the mean convexity of M is equivalent to the σ_1 convexity of M.

For mean curvature flows of mean curvature sets $\{F_t(K)\}_{t\geq 0}$ (e.g. $K_0 = F_0(\partial K)$ is a mean convex smooth hypersurface), a striking and difficult theorem by White [W3] (theorem 1) claimed that the singular set of $\mathcal{K} = \{(x,t) : x \in F_t(\partial K), t \geq 0\}$, sing($\mathcal{K}$), has parabolic Hausdorff dimension at most $n-1$. Here the singular set is defined to the completment of these regular points near where $\mathcal{K}$ is a smooth manifold and has its tangent plane non-horizontal. As a direct consequence of this regularity theorem of White, one knows that the top dimensional subset $\Sigma_n \setminus \Sigma_{n-1} \subset M_T$ defined as in the proposition B is regular set of M_T, namely near each point in $\Sigma_n \setminus \Sigma_{n-1}$ M_T is a smooth n-dimensional manifold.

By exploring the estimates (1.6) on σ_k curvatures for the mean curvature flow of mean convex hypersurfaces (1.1)-(1.2) obtained by [HS2] and the partial regularity theorem for mean convex flows of [W3] (i.e. the singular set has parabolic Hausdorff dimension at most $n-1$), we obtain the following result.

Theorem C. *Assume that the Brakke flow $\{u_t\}_{t\in R_+}$ is given as same as that in Proposition B. For any $2 \leq k \leq n-1$, if the initial closed hypersurface M_0 is σ_k convex. Then*

$$\Sigma_{n-1} \setminus \Sigma_{n-2} = \cdots = \Sigma_{n-k+1} \setminus \Sigma_{n-k} = \emptyset. \tag{1.11}$$

This note is written as follows. In §2, we prove Theorem A; In §3, we give a proof of Proposition B; In §4, we prove Theorem C.

§2 Proof of Theorem A

This section is devoted to the proof of theorem A. First, we show

Lemma 2.1. *M_t converges to M_T in the Hausdorff distance sense, as $t \uparrow T$.*

Proof. For any $t \uparrow T$, we can extract a subsequence $t_i \uparrow T$ and a closed subset $A \subset R^{n+1}$ such that M_{t_i} converges to A in the Hausdorff distance. Now we want to show $A = M_T$. Suppose $x_0 \notin A$. Then there exists $r_0 > 0$ such that $B_{r_0}(x_0) \cap M_{t_i} = \emptyset$ for i sufficiently large. In particular, $H^n(M_{t_i} \cap B_{r_0}(x_0)) = 0$. Hence, $\mu_T(B_{r_0}(x_0)) = 0$ and $x_0 \notin M_T$. This gives that $M_T \subset A$. To prove

$A \subset M_T$, we argue by contradiction. Suppose that there exists $x_0 \in A \setminus M_T$. Then there exists $r_0 > 0$ such that $\mu_T(B_{r_0}(x_0)) = 0$. On the other hand, there exist $x_i \in M_{t_i}$ such that $x_i \to x_0$. Therefore, by the upper semicontinuity of $\Theta(\mu, \cdot)$ and the fact that $\Theta(\mu, (x_i, t_i)) = 1$ (since $x_i \in M_{t_i}$ and M_{t_i} is smooth), we have

$$\Theta(\mu, (x_0, T)) \geq \limsup_{i \to \infty} \Theta(\mu, (x_i, t_i)) = 1.$$

This implies that $x_0 \in M_T$ (for otherwise $\Theta(\mu, (x_0, T)) = 0$). We get the desired contradiction. ■

Proof of Theorem A. Since for any $0 \leq t < T$ and $y \in M_t$, one has $(\Theta, (y, t)) = 1$. The upper semicontinuity implies that $\Theta(\mu, (x_0, T)) \geq 1$ for any $x_0 \in M_T$. For any $\epsilon > 0$, the estimate of Cheng [C] implies that there exists sufficiently large $K_\epsilon > 0$ such that

$$\begin{aligned} G_{(x_0,T)}(x, T - r^2) &\leq r^{-n}, \qquad\qquad \forall x \in R^{n+1}, \\ &\leq \epsilon G_{(x_0, T+r^2)}(x, T - r^2), \forall |x - x_0| \geq K_\epsilon r. \end{aligned}$$

Therefore, we have, for any $x_0 \in M_T$,

$$\begin{aligned} 1 &\leq \Theta(\mu, (x_0, T)) \\ &\leq C r^{-n} H^n(M_{T-r^2} \cap B_{K_\epsilon r}(x_0)) \\ &+ \epsilon \int_{M_{T-r^2}} G_{(x_0, T+r^2)}(x, T - r^2). \end{aligned} \tag{2.1}$$

The monotonicity for $\Theta(\mu, (x_0, T + r^2), \cdot)$ implies that

$$\begin{aligned} &\int_{M_{T-r^2}} G_{(x_0,T+r^2)}(x, T - r^2) \\ &= \int_{M_{T+r^2-2r^2}} G_{(x_0,T+r^2)}(x, T + r^2 - 2r^2) \\ &\leq \int_{M_{T+r^2-R^2}} G_{(x_0.T+r^2)}(x, T + r^2 - R^2) \\ &\leq C R^{-n} H^n(M_{T+r^2-R^2}) \leq C R^{-n} H^n(M_0) \end{aligned}$$

for any $\sqrt{2} r \leq R \leq \sqrt{T + r^2}$. In particular, for sufficiently small r, by choosing $R^2 = \frac{T}{2}$, we have

$$\int_{M_{T-r^2}} G_{(x_0,T+r^2)}(x, T - r^2) \leq C T^{-\frac{n}{2}} H^n(M_0).$$

Hence, by choosing $\epsilon = \epsilon(M_0, T) > 0$ sufficiently small, we have

$$r^n \leq CH^n(M_{T-r^2} \cap B_{K_\epsilon r}(x_0)). \tag{2.2}$$

Observe that the family $\mathcal{F} = \{B_{K_\epsilon r}(x) : x \in M_T\}$ covers M_T so that the Vitali's covering Lemma implies that there exists a disjoint subfamily $\{B_{K_\epsilon r}(x_i) : x_i \in M_T\}_{i=1}^\infty$ such that

$$M_T \subset \cup_i B_{5K_\epsilon r}(x_i),$$

so that

$$\begin{aligned} H^n_{5K_\epsilon r}(M_T) &\leq (5K_\epsilon)^n \sum_i r^n \\ &\leq (5K_\epsilon)^n \sum_i H^n(M_{T-r^2} \cap B_{K_\epsilon r}(x_i)) \\ &= (5K_\epsilon)^n H^n(M_{T-r^2} \cap (\cup_i B_{K_\epsilon r}(x_i))) \\ &\leq (5K_\epsilon)^n H^n(M_{T-r^2}) \leq C(\epsilon, M_0) < \infty. \end{aligned}$$

This finishes the proof for Theorem A. ■

§3 Proof of Proposition B

In this section, we give a slightly different and also easier proof of Proposition B, which is essentially due to White [W1].

For any $0 \leq i \leq n$, it follows from the definition of Σ_i that for any $x_0 \in \Sigma_i$ and each tangent flow $\tilde{\mu}$ of μ at (x_0, T), which is a backwardly self-similar Brakke flow, $V(\tilde{\mu}) = \{x \in R^{n+1} : \Theta(\tilde{\mu}, (x, 0)) = \Theta(\tilde{\mu}, (0, 0))\}$ is a vector space of dimension at most i. Moreover, it follows from the argument in [W1] (see also theorem 3 [W2]) that if we let $W(\tilde{\mu}) \equiv \{(x,t) \in R^{n+1} \times R_- : \Theta(\tilde{\mu}, (x,t)) = \Theta(\tilde{\mu}, (0,0))\}$ then either $W(\tilde{\mu}) = V(\tilde{\mu}) \times R_-$ and there exists a minimal hypercone $C^{n-d} \in R^{n+1-d}$ such that

$$\tilde{\mu}|_{t \in R_-} = (R^d \times C^{n-d}) \times R_-, \tag{3.1}$$

or $W(\tilde{\mu}) = V(\tilde{\mu})$ and there exists a backwardly self-similar $(n-d)$ dimensional Brakke flow $\nu = \{\nu_t\}_{t \in R}$ in R^{n+1-d} such that

$$\tilde{\mu}|_{t \in R_-} = V(\tilde{\mu}) \times \nu|_{t \in R_-}, \tag{3.2}$$

here $d = \dim(V(\tilde{\mu}))$. Define

$$\eta_{x,\rho}(y) = \rho^{-1}(y - x), \forall y \in R^{n+1}$$

Now we claim

Claim 3.1. *For any $x_0 \in \Sigma_i$ and each $\delta > 0$ there exists an $\epsilon > 0$ (depending on μ, x_0, δ) such that for each $\rho \in (0, \epsilon]$*

$$\begin{aligned} &\eta_{x_0,\rho}\{x \in B_\rho(x_0) : \Theta(\mu, (x, T)) \geq \Theta(\mu, (x_0, T)) - \epsilon\} \\ &\subset \ \textit{the } \delta - \textit{neighbourhood of } L_{x_0,\rho} \end{aligned} \tag{3.3}$$

for some i-dimensional subspace $L_{x_0,\rho}$ of R^{n+1}.

Proof. If this is false, there exist $\delta > 0$ and $x_0 \in \Sigma_i$ and $\rho_k \downarrow 0$ and $\epsilon_k \downarrow 0$ such that

$$\begin{aligned} &\{x \in B_1(0) : \Theta(P_{(x_0,T),\rho_k}(\mu), (x, 0)) \geq \Theta(\mu, (x_0, T)) - \epsilon_k\} \\ &\not\subset \delta - \text{neighbourhood of } L, \end{aligned} \tag{3.4}$$

for any i-dimensional subspace L of R^{n+1}. But $P_{(x_0,T),\rho_k}(\mu) \to \tilde{\mu}$, a tangent flow of μ at (x_0, T), and $\Theta(\tilde{\mu}, (0, 0)) = \Theta(\mu, (x_0, T))$. Since $x_0 \in \Sigma_i$, we have $\dim(V(\tilde{\mu})) \leq i$, there is a i-dimensional subspace $L_0 \subset R^{n+1}$ such that $V(\tilde{\mu}) \subset L_0$. Moreover, the uppersemicontinuity of $\Theta(\tilde{\mu}, \cdot)$ implies that there is a $\alpha > 0$ such that

$$\Theta(\tilde{\mu}, (x, 0)) < \Theta(\tilde{\mu}, (0, 0)) - \alpha, \forall x \in B_1(0) \text{with } \mathrm{dist}(x, L_0) \geq \delta. \tag{3.5}$$

Then the upper semicontinuity of $\Theta(\mu, \cdot)$ for convergence of both of its variables implies that we must have, for k' sufficiently large,

$$\Theta(P_{(x_0,T),\rho_{k'}}(\mu), (x, 0)) < \Theta(\tilde{\mu}, (0, 0)) - \alpha, \forall x \in B_1(0) \text{ with } \mathrm{dist}(x, L_0) \geq \delta. \tag{3.6}$$

This clearly contradicts with (3.4). The claim is proven. ■

Completion of Proof of Proposition B. We decompose $\Sigma_i = \cup_{l=1}^{\infty} \Sigma_i^l$, where Σ_i^l denotes the points $x \in \Sigma_i$ such that the claim 3.1 holds for $\epsilon = l^{-1}$. Now we

decompose $\Sigma_i^l = \cup_{q=1}^{\infty} \Sigma_i^{l,q}$, where $\Sigma_i^{l,q} = \{x \in \Sigma_i^l : \frac{q-1}{i} \leq \Theta(\mu, x) \leq \frac{q}{i}\}$. Hence, claim 3.1 implies that for $A = \Sigma_i^{l,q}$,

$$\eta_{x,\rho}(A \cap B_\rho(x)) \subset \delta - \text{neighbourhood of } L_{x,\rho}, \forall x \in A, \rho < l^{-1}. \tag{3.7}$$

for some i-dimensional subspace $L_{x,\rho} \subset R^{n+1}$. The proof is completed if we apply the following Lemma, whose proof can be found in the Lecture 2.4 of Simon [S].

Lemma 3.2. *There is a $\beta : R_+ \to R_+$ with $\lim_{t \downarrow 0} \beta(t) = 0$ such that if $\delta > 0$ and if $A \in R^{n+1}$ satisfying the property (3.7) above, then $H^{i+\beta(\delta)}(A) = 0$.*

■

§4 Proof of Theorem C

In this section, we outline the proof of the theorem C. But, first, we gather together needed key estimates by [HS2] on the σ_k curvature under the mean curvature flow (1.1)-(1.2) with the initial M_0 being mean convex.

Lemma 4.1. *(a). For any closed hypersurface M in R^{n+1}. For any $1 \leq k \in n$, if M is σ_k convex (or σ_k strictly convex, respectively) then M is also σ_l convex (or σ_l strictly convex, respectively) for all $1 \leq l \leq k$. In particular, M is mean convex.*

(b). For any $1 \leq k \leq n$. Assume that M_0 is a σ_k convex closed hypersurface and $\{M_t\}_{0 \leq t < T}$ is the mean curvature flow (1.1)-(1.2). Then, for any $0 < t < T$, M_t is σ_k strictly convex. In particular, M_t is σ_l strictly convex for $0 < t < T$ and all $1 \leq l \leq k$.

Proof. The reader can find the details of the proof in Proposition 3.3 (i) (ii) in [HS2]. ■

It follows from Lemma Lemma 4.1 that for $1 \leq k \leq n$ if M_0 is a σ_k convex closed hypersurface then for any $0 < t < T$ there exists a $\epsilon = \epsilon_t > 0$ such that

$$\sigma_l^{M_t}(x) \geq \epsilon H^l(x,t), \forall x \in M_t, 1 \leq l \leq k. \tag{4.1}$$

Proposition 3.4 of [HS2] then asserts that the inequality (4.1) is kept under the mean curvature flow (1.1)-(1.2). More precisely, we have

Lemma 4.2. *For $1 \le k \le n$. Let $\{M_t\}_{0 \le t < T}$ be the mean curvature flow (1.1)-(1.2), with M_0 being a σ_k convex closed hypersurface. Then (3.1) holds with $\epsilon = \epsilon_t > 0$ for all $s \in [t, T)$ and all $1 \le l \le k$.*

Proof of Theorem C. Suppose that the conclusion fails. Then there exists a $2 \le i \le k$ such that $\Sigma_{n-i+1} \setminus \Sigma_{n-i} \neq \emptyset$. Pick a point $x_0 \in \Sigma_{n-i+1} \setminus \Sigma_{n-i}$. Then it follows from the definition that any tangent flow of μ at (x_0, T) has no more than $n - i + 1$ dimension of spatial translating invariant directions, and there exists at least one $\lambda_m \downarrow 0$ and a vector subspace $V \subset R^{n+1}$, with $\dim(V) = n - i + 1$, such that either (3.1) or (3.2) holds, namely either (a): there exists a minimal hypercone $C^{i-1} \subset R^i$ such that

$$P_{(x_0,T),\lambda_m}(\mu)|_{t \in R_-} \to (V \times C^{i-1}) \times R_-, \tag{4.1}$$

or (b): there exists a backwardly self-similar $(i-1)$ dimensional Brakke flow $\nu = \{\nu_t\}_{t \in R}$ in R^i such that

$$P_{(x_0,T),\lambda_m}(\mu)|_{t \in R_-} \to V \times \nu|_{t \in R_-}. \tag{4.2}$$

As a special case of the partial regularity theorem of White [W1], we know that the singular set of both C^{i-1} (in the case (a)) and $\mathcal{M}_{-1} = \text{spt}(V \times \nu_{-1})$ (in the case (b)) has Hausdorff dimension at most $i - 4$, here the singular set of a subset $A \in R^i$ is the completment of regular points in A and the regular points are points near where A is a smooth manifold. For, otherwise, the singular set of the corresponding tangent flow has parabolic Hausdorff dimension larger than $(i-4) + (n-i+1) + 2 = n - 1$, which contradicts with White's theorem. Since case (a) can be handled similarly to case (b), we want to discuss case (b) only. Let $N \subset \mathcal{M}_{-1}$ denotes the singular set. Then, for any $x \in \mathcal{M}_{-1} \setminus N$, $V \times \nu$ is smooth in a spacetime neighbourhood U of $(x, -1)$. Moreover, by the Brakke (unit density) regularity theorem ([B]), we can assume that $\mu_m \equiv P_{(x_0,T),\lambda_m}(\mu)$ converge smoothly to $V \times \nu$ in the neighbourhood U. We now claim

Claim 4.3. *The second fundamental form A of $\mathcal{M}_{-1}$ vanishes everywhere on $\mathcal{M}_{-1} \setminus N$.*

Proof of Claim 4.3. Since M_0 is assumed to be σ_k convex, it follows from part (b) of Lemma 4.1 that without loss of generality we can assume that M_0 is in fact σ_k strictly convex so that Lemma 4.2 implies that there exists a $\epsilon > 0$ such that

$$\sigma_l^{M_t}(x) \geq \epsilon H^l(x,t) > 0, \forall x \in M_t, 0 \leq t < T, 1 \leq l \leq k. \tag{4.3}$$

Note that (4.3), combined with scalings, implies that, for any $1 \leq l \leq k$, $t \in [-\lambda_m^{-2}T_0, 0)$, and $x \in \mathrm{spt}(\mu_m(t)) = \mathrm{spt}(\mu_m \cap \{t\})$,

$$\begin{aligned} \sigma_l^{\mu_m(t)}(x) &= \lambda_m^l \sigma_l^{M_{T_0+\lambda_m^2 t}}(x_0 + \lambda_m x) \\ &\geq \epsilon \lambda_m^l H^l(x_0 + \lambda_m x, T_0 + \lambda_m^2 t) \\ &= \epsilon(\sigma_1^{\mu_m(t)}(x))^l, \end{aligned} \tag{4.4}$$

$$H(x,t) > 0, \forall x \in \mathrm{spt}(\mu_m(t)), t \in [\lambda_m^{-2}T_0, 0). \tag{4.5}$$

This, combined with the smooth convergence fact at $\mathcal{M}_{-1} \setminus N$ as above, implies, for any $1 \leq l \leq k$

$$\sigma_l^{\mathcal{M}_{-1}}(x) \geq \epsilon H^l(x,-1) \geq 0, \forall x \in \mathcal{M}_{-1} \setminus N. \tag{4.6}$$

On the other hand, since $\mathcal{M}_{-1} = V \times \mathrm{spt}(\nu_{-1})$ and $\dim(V) = n - i + 1$, it follows from the definition 1.1 that

$$\sigma_i^{\mathcal{M}_{-1}}(x) = \cdots = \sigma_k^{\mathcal{M}_{-1}}(x) = 0, \forall x \in \mathcal{M}_{-1} \setminus N. \tag{4.7}$$

(4.6) and (4.7) imply that

$$H(x,-1) = 0, \forall x \in \mathcal{M}_{-1} \setminus N. \tag{4.8}$$

Therefore, for any $x \in \mathcal{M}_{-1} \setminus N$,

$$\begin{aligned} \sigma_2^{\mathcal{M}_{-1}}(x) &= \frac{1}{2}(H^2(x,-1) - |A|^2(x,-1)) \\ &= -\frac{1}{2}|A|^2(x,-1) \leq 0. \end{aligned}$$

This, combined with (4.6), implies that

$$\sigma_2^{\mathcal{M}_{-1}}(x) = |A|(x,-1) = 0, \forall x \in \mathcal{M}_{-1} \setminus N. \tag{4.9}$$

This finishes the proof of Claim 4.3. It follows from the Claim 4.3 that $\mathcal{M}_{-1} \setminus N$ is flat. Note also that $V \times \nu_t|_{t \in R_-}$ is also a backwardly self-similar Brakke flow. Therefore, we know that $\mathcal{M}_{-1}$ is also a minimal hypercone in R^{n+1}. Hence $\mathcal{M}_{-1}$ is a hyperplane in R^{n+1}. This clearly contradicts with the fact that $2 \leq i \leq k$ and $x_0 \in \Sigma_{n-i+1} \setminus \Sigma_{n-i}$. Therefore, the proof is complete. ■

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